Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 397–406

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1997

MODIFIED HADAMARD PRODUCT PROPERTIES OF CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY RUSCHEWEYH DERIVATIVE

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Received 08 May, 2016

Abstract. In this paper we study the modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative.
The obtained results are sharp and they improve known results.

2010 *Mathematics Subject Classification:* 30C45

Keywords: analytic functions, modified Hadamard product, Ruscheweyh derivative

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

Ruscheweyh [5] defined that

$$D^{\gamma} f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1) \tag{1.3}$$

which implies

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \left(n \in \mathbb{N}_0 = \{0, 1, 2 \ldots\}\right). \tag{1.4}$$

The symbol $D^n f(z) (n \in \mathbb{N}_0)$ was called the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. It is easy to see that

$$D^0 f(z) = f(z), D^1 f(z) = z f'(z)$$

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k \tag{1.5}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.6}$$

Definition 1. Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z))$; $\forall z \in U$. We denote by $\prec$ the subordination relation.

Definition 2. [2,4] For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $Q(n,\lambda,A,B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form (1.1) such that

$$(1-\lambda)(D^n f(z))' + \lambda(D^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{1.7}$$

Definition 3 ([7]). A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_k)$ if $f \in A$ and $arg(a_k) = \theta_k$,$\forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that $\theta_k + (k-1)\delta \equiv \pi(mod 2\pi), \forall k \geq 2$ then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V. Let $VQ(n,\lambda,A,B)$ denote the subclass of V consisting of functions $f(z) \in Q(n,\lambda,A,B)$.

Theorem 1 ([4]). *Let the function $f(z)$ defined by (1.1) be in V. Then $f(z) \in VQ(n,\lambda,A,B)$, if and only if*

$$\sum_{k=2}^{\infty} k\delta(n,k) C_k |a_k| \leq (B-A)(n+1) \tag{1.8}$$

where

$$C_k = (1+B)[n+1+\lambda(k-1)].$$

The extremal functions are

$$f_k(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)} e^{i\theta_k} z^k, (k \geq 2).$$

The modified Hadamard product of two functions f and g of the form (1.1) and (1.2), and which belong to $V(\theta_k, \delta)$ is defined by (see also [3,6,8])

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.9}$$

2. MAIN RESULTS

Theorem 2. *If* $f \in VQ(n,\lambda,A_1,B), g \in VQ(n,\lambda,A_2,B)$ *then* $f * g \in VQ(n,\lambda,A^*,B)$, *where* $A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2\delta(n,2)}$.
The result is sharp.

Proof. Let $f \in VQ(n,\lambda,A_1,B), g \in VQ(n,\lambda,A_2,B)$ and suppose they have the form (1.1). Since $f \in VQ(n,\lambda,A_1,B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k|}{B-A_1} \leq n+1 \tag{2.1}$$

and for $g \in VQ(n,\lambda,A_2,B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,|b_k|}{B-A_2} \leq n+1. \tag{2.2}$$

We know from Theorem 1 that $f * g \in VQ(n,\lambda,A^*,B)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k b_k|}{B-A^*} \leq n+1. \tag{2.3}$$

By using the Cauchy-Schwarz inequality for (2.1) and (2.2) we have

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}} \leq n+1.$$

We note that

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k b_k|}{B-A^*} \leq \frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

implies (2.3). But this is equivalent to

$$\frac{|a_k b_k|}{B-A^*} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B-A^*}{\sqrt{(B-A_1)(B-A_2)}}. \tag{2.4}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{(B-A_1)(n+1)}{kC_k\delta(n,k)} \text{ and } |b_k| \leq \frac{(B-A_2)(n+1)}{kC_k\delta(n,k)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B-A_1)(B-A_2)}(n+1)}{kC_k\delta(n,k)}, (k \geq 2). \tag{2.5}$$

From (2.5) we obtain that (2.4) holds if

$$\frac{\sqrt{(B-A_1)(B-A_2)}(n+1)}{kC_k\delta(n,k)} \leq \frac{B-A^*}{\sqrt{(B-A_1)(B-A_2)}}$$

or equivalently

$$A^* \leq B - \frac{(B-A_1)(B-A_2)(n+1)}{kC_k\delta(n,k)}.$$

But $kC_k\delta(n,k) < (k+1)C_{k+1}\delta(n,k+1), (k \geq 2)$ so

$$B - \frac{(B-A_1)(B-A_2)(n+1)}{kC_k\delta(n,k)} \geq B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2\delta(n,2)}, (k \geq 2)$$

$$\Rightarrow A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2\delta(n,2)}.$$

The result is sharp, because if

$$f(z) = z + \frac{(B-A_1)(n+1)}{2C_2\delta(n,2)} e^{i\theta_1} z^2 \in VQ(n,\lambda,A_1,B)$$

$$g(z) = z + \frac{(B-A_2)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2 \in VQ(n,\lambda,A_2,B)$$

$$f * g \in VQ(n,\lambda,A^*,B)$$

and satisfy (8) with equality. Indeed,

$$2\delta(n,2)C_2 \frac{(B-A_1)(B-A_2)(n+1)^2}{2^2 C_2^2 [\delta(n,2)]^2} = (B-A^*)(n+1)$$

because

$$B - A^* = \frac{(B-A_1)(B-A_2)(n+1)}{2C_2\delta(n,2)}.$$

□

Corollary 1. *If $f,g \in VQ(n,\lambda,A,B)$ then $f * g \in VQ(n,\lambda,A^*,B)$, where $A^* = B - \frac{(B-A)^2(n+1)}{2C_2\delta(n,2)}$. The result is sharp.*

Theorem 3. *If $f \in VQ(n,\lambda,A,B_1), g \in VQ(n,\lambda,A,B_2)$ then $f * g \in VQ(n,\lambda,A,B^*)$, where $B^* = A + \frac{(B_1-A)(B_2-A)(n+1)}{2C_2\delta(n,2)}$. The result is sharp.*

Proof. Let $f \in VQ(n,\lambda,A,B_1), g \in VQ(n,\lambda,A,B_2)$ and suppose they have the form (1.1). Since $f \in VQ(n,\lambda,A,B_1)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k|}{B_1 - A} \leq n+1 \tag{2.6}$$

and for $g \in VQ(n,\lambda,A,B_2)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|b_k|}{B_2 - A} \leq n+1. \tag{2.7}$$

We know from Theorem 1 that $f * g \in VQ(n,\lambda,A,B^*)$ if and only if

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k b_k|}{B^* - A} \leq n+1. \tag{2.8}$$

By using the Cauchy-Schwarz inequality for (2.6) and (2.7) we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,\sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}} \leq n+1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k b_k|}{B^* - A} \leq \frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,\sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}}$$

implies (2.8). But this is equivalent to

$$\frac{|a_k b_k|}{B^* - A} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B^* - A}{\sqrt{(B_1 - A)(B_2 - A)}}. \tag{2.9}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{(B_1 - A)(n+1)}{kC_k\delta(n,k)} \text{ and } |b_k| \leq \frac{(B_2 - A)(n+1)}{kC_k\delta(n,k)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B_1 - A)(B_2 - A)}(n+1)}{kC_k\delta(n,k)}, (k \geq 2). \tag{2.10}$$

from (2.10) we obtain that (2.9) holds if

$$\frac{\sqrt{(B_1-A)(B_2-A)}(n+1)}{kC_k\delta(n,k)} \le \frac{B^*-A}{\sqrt{(B_1-A)(B_2-A)}}$$

or equivalently

$$B^* \ge A + \frac{(B_1-A)(B_2-A)(n+1)}{kC_k\delta(n,k)}.$$

But $kC_k\delta(n,k) < (k+1)C_{k+1}\delta(n,k+1), (k \ge 2)$ so :

$$A + \frac{(B_1-A)(B_2-A)(n+1)}{kC_k\delta(n,k)} \le A + \frac{(B_1-A)(B_2-A)(n+1)}{2C_2\delta(n,2)}, (k \ge 2)$$

$$\Rightarrow B^* = A + \frac{(B_1-A)(B_2-A)(n+1)}{2C_2\delta(n,2)}.$$

The result is sharp, because if

$$f(z) = z + \frac{(B_1-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_1} z^2 \in VQ(n,\lambda,A,B_1)$$

$$g(z) = z + \frac{(B_2-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2 \in VQ(n,\lambda,A,B_2)$$

$$f * g \in VQ(n,\lambda,A,B^*)$$

and satisfy (8) with equality. Indeed,

$$2\delta(n,2)C_2 \frac{(B_1-A)(B_2-A)(n+1)^2}{2^2C_2^2[\delta(n,2)]^2} = (B^*-A)(n+1)$$

because

$$B^* - A = \frac{(B_1-A)(B_2-A)(n+1)}{2C_2\delta(n,2)}.$$

□

Corollary 2. *If $f,g \in VQ(n,\lambda,A,B)$ then $f * g \in VQ(n,\lambda,A,B^*)$, where $B^* = A + \frac{(B-A)^2(n+1)}{2C_2\delta(n,2)}$. The result is sharp.*

Theorem 4. *If $f_j \in VQ(n,\lambda,A_j,B), j = \overline{1,s}, s \in \{2,3,4,\dots\}$ then $f_1 * f_2 * \dots * f_s \in VQ(n,\lambda,A^{(s-1)*},B)$, where $A^{(s-1)*} = B - \frac{(n+1)^{s-1}\prod_{j=1}^{s}(B-A_j)}{2^{s-1}C_2^{s-1}[\delta(n,2)]^{s-1}}$. The result is sharp.*

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1.1).
Let $s = 2$. If $f_j \in VQ(n,\lambda,A_j,B), j = \overline{1,2}$ then $f_1 * f_2 \in VQ(n,\lambda,A^*,B)$ where $A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{kC_k\delta(n,k)}$, from Theorem 2 is true.

Assume, for $s = m$, that the formula displayed below holds.
If $f_j \in VQ(n,\lambda,A_j,B)$, $j = \overline{1,m}, m \in \{2,3,4,\dots\}$ then
$f_1 * f_2 * \dots * f_m \in VQ(n,\lambda,A^{(m-1)*},B)$, where
$A^{(m-1)*} = B - \frac{(n+1)^{m-1}\prod_{j=1}^{m}(B-A_j)}{k^{m-1}C_k^{m-1}[\delta(n,k)]^{m-1}}$.

Let $s = m+1$: if $f_1 * f_2 * \dots * f_m \in VQ(n,\lambda,A^{(m-1)*},B), m \in \{2,3,4,\dots\}$ and $f_{m+1} \in VQ(n,\lambda,A_{m+1},B)$ then we have to prove

$f_1 * f_2 * \dots * f_m * f_{m+1} \in VQ(n,\lambda,A^{m*},B)$, where $A^{m*} = B - \frac{(n+1)^m \prod_{j=1}^{m+1}(B-A_j)}{k^m C_k^m[\delta(n,k)]^m}$.

For the proof we use the result of Theorem 2:

$$A^{m*} \le B - \frac{(B-A^{(m-1)*})(B-A_{m+1})(n+1)}{kC_k\delta(n,k)}$$

$$A^{m*} \le B - \frac{\frac{(n+1)^{m-1}\prod_{j=1}^{m}(B-A_j)}{k^{m-1}C_k^{m-1}[\delta(n,k)]^{m-1}}(B-A_{m+1})(n+1)}{kC_k\delta(n,k)}$$

$$A^{m*} \le B - \frac{(n+1)^m \prod_{j=1}^{m+1}(B-A_j)}{k^m C_k^m\,[\delta(n,k)]^m}.$$

But $kC_k\delta(n,k) < (k+1)C_{k+1}\delta(n,k+1), (k \ge 2)$ so:

$$B - \frac{(n+1)^m \prod_{j=1}^{m+1}(B-A_j)}{k^m C_k^m\,[\delta(n,k)]^m} \ge B - \frac{(n+1)^m \prod_{j=1}^{m+1}(B-A_j)}{2^m C_2^m\,[\delta(n,2)]^m}, (k \ge 2) \Rightarrow$$

$$A^{m*} = B - \frac{(n+1)^m \prod_{j=1}^{m+1}(B-A_j)}{2^m C_2^m\,[\delta(n,2)]^m}.$$

The result is sharp, because if

$$f(z) = z + \frac{(B-A^{(s-1)*})(n+1)}{2C_2\delta(n,2)}e^{i\theta_1}z^2 \in VQ(n,\lambda,A^{(s-1)*},B)$$

$$g(z) = z + \frac{(B-A_s)(n+1)}{2C_2\delta(n,2)}e^{i\theta_2}z^2 \in VQ(n,\lambda,A_s,B)$$

$$f * g \in VQ(n,\lambda,A^{s*},B)$$

and satisfy (8) with equality. Indeed,

$$2\delta(n,2)C_2\frac{(B-A^{(s-1)*})(B-A_s)(n+1)^2}{2^2C_2^2\,[\delta(n,2)]^2}=(B-A^{s*})(n+1)$$

because

$$B-A^{s*}=\frac{(B-A^{(s-1)*})(B-A_s)(n+1)}{2C_2\delta(n,2)}\Leftrightarrow$$

$$\Leftrightarrow B-A^{s*}=\frac{(n+1)^s\prod\limits_{j=1}^{s+1}(B-A_j)}{2^sC_2^s\,[\delta(n,2)]^s}.$$

□

Theorem 5. *If* $f_j\in VQ(n,\lambda,A,B_j),\,j=\overline{1,s},s\in\{2,3,4,\ldots\}$ *then* $f_1*f_2*\ldots*f_s\in VQ(n,\lambda,A,B^{(s-1)*})$*, where* $B^{(s-1)*}=A+\frac{(n+1)^{s-1}\prod\limits_{j=1}^{s}(B_j-A)}{2^{s-1}C_2^{s-1}[\delta(n,2)]^{s-1}}$. *The result is sharp.*

Proof. For the proof we use the mathematical induction method and suppose that $f_j,\forall j$ have the form (1.1).
Let $s=2$. If $f_j\in VQ(n,\lambda,A,B_j),\,j=\overline{1,2}$ then $f_1*f_2\in VQ(n,\lambda,A,B^*)$ where $B^*=A+\frac{(B_1-A)(B_2-A)(n+1)}{kC_k\delta(n,k)}$, from Theorem 3 is true.
Assume, for $s=m$, that the formula displayed below holds.
If $f_j\in VQ(n,\lambda,A,B_j),\,j=\overline{1,m},m\in\{2,3,4,\ldots\}$ then

$f_1*f_2*\ldots*f_m\in VQ(n,\lambda,A,B^{(m-1)*})$, where $B^{(m-1)*}=A+\frac{(n+1)^{m-1}\prod\limits_{j=1}^{m}(B_j-A)}{k^{m-1}C_k^{m-1}[\delta(n,k)]^{m-1}}$.

Let $s=m+1$: if $f_1*f_2*\ldots*f_m\in VQ(n,\lambda,A,B^{(m-1)*}),m\in\{2,3,4,\ldots\}$ and $f_{m+1}\in VQ(n,\lambda,A,B_{m+1})$ then we have to prove

$f_1*f_2*\ldots*f_m*f_{m+1}\in VQ(n,\lambda,A,B^{m*})$, where $B^{m*}=A+\frac{(n+1)^m\prod\limits_{j=1}^{m+1}(B_j-A)}{k^mC_k^m[\delta(n,k)]^m}$.

For the proof we use the result of Theorem 3:

$$B^{m*}\ge A+\frac{(B^{(m-1)*}-A)(B_{m+1}-A)(n+1)}{kC_k\delta(n,k)}$$

$$B^{m*}\ge A+\frac{\frac{(n+1)^{m-1}\prod\limits_{j=1}^{m}(B_j-A)}{k^{m-1}C_k^{m-1}[\delta(n,k)]^{m-1}}(B_{m+1}-A)(n+1)}{kC_k\delta(n,k)}$$

$$B^{m*}\ge A+\frac{(n+1)^m\prod\limits_{j=1}^{m+1}(B_j-A)}{k^mC_k^m\,[\delta(n,k)]^m}.$$

But $kC_k\delta(n,k) < (k+1)C_{k+1}\delta(n,k+1), (k \geq 2)$ so :

$$A + \frac{(n+1)^m \prod_{j=1}^{m+1}(B_j - A)}{k^m C_k^m [\delta(n,k)]^m} \leq A + \frac{(n+1)^m \prod_{j=1}^{m+1}(B_j - A)}{2^m C_2^m [\delta(n,2)]^m}, (k \geq 2)$$

$$\Rightarrow B^{m*} = A + \frac{(n+1)^m \prod_{j=1}^{m+1}(B_j - A)}{2^m C_2^m [\delta(n,2)]^m}.$$

The result is sharp, because if

$$f(z) = z + \frac{(B^{(s-1)*} - A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_1} z^2 \in VQ(n,\lambda,A,B^{(s-1)*})$$

$$g(z) = z + \frac{(B_s - A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2 \in VQ(n,\lambda,A,B_s)$$

$$f * g \in VQ(n,\lambda,A,B^{s*})$$

and satisfy (8) with equality. Indeed,

$$2\delta(n,2)C_2 \frac{(B^{(s-1)*} - A)(B_s - A)(n+1)^2}{2^2 C_2^2 [\delta(n,2)]^2} = (B^{s*} - A)(n+1)$$

because

$$B^{s*} - A = \frac{(B^{(s-1)*} - A)(B_s - A)(n+1)}{2C_2\delta(n,2)} \Leftrightarrow$$

$$\Leftrightarrow B^{s*} - A = \frac{(n+1)^s \prod_{j=1}^{s+1}(B_j - A)}{2^s C_2^s [\delta(n,2)]^s}.$$

□

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